

# THERMODYNAMICAL FORMALISM ASSOCIATED WITH INDUCING SCHEMES FOR ONE-DIMENSIONAL MAPS

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ABSTRACT. For a smooth map  $f$  of a compact interval  $I$  admitting an inducing scheme we establish a thermodynamical formalism, i.e., describe a class of real-valued potential functions  $\varphi$  on  $I$  which admit a unique equilibrium measure  $\mu_\varphi$ . Our results apply to unimodal maps corresponding to a positive Lebesgue measure set of parameters in a one-parameter transverse family.

## 1. INTRODUCTION

In this note we describe some results on thermodynamics for a class of smooth one-dimensional maps. In the classical thermodynamical formalism, given a continuous map  $f$  of a compact metric space  $I$  and a continuous real valued *potential* function  $\varphi$  on  $I$ , one studies the *equilibrium* measures for  $\varphi$ , i.e., invariant Borel probability measures  $\mu$  on  $I$  for which the supremum

$$\sup \left\{ h_\mu(f) + \int_I \varphi d\mu \right\}$$

is attained, where  $h_\mu(f)$  denotes the metric entropy of the map  $f$ . In this paper we are interested in the existence, uniqueness and ergodic properties of equilibrium measures for a smooth one-dimensional map that admits an "inducing scheme" as described below. It represents  $f$ , restricted to a subset  $X \subset I$ , as a *tower*  $(W, \tau, F)$  where  $F$  is the *induced map* acting on the *inducing domain*  $W \subset I$  and the *inducing time*  $\tau$  is a positive integer-valued function on  $W$ . The latter is usually not the first return time to  $W$ . An important feature of the inducing scheme is that  $W$  admits a countable generating Bernoulli partition. Thus,  $F$  is equivalent to the full shift on a countable set of states.

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We apply results of Mauldin and Urbański [MU01] and of Sarig [Sar03], [Sar99] (see also Aaronson, Denker and Urbanski [ADU93], Yuri [Yur99] and Buzzi and Sarig [BS03]) to establish the existence and uniqueness of equilibrium measures for the induced map  $F$ . We then lift them from the inducing domain to the tower. The latter procedure is quite subtle and makes use of the integrability of the inducing time with respect to the equilibrium measures. It also requires a relation between  $F$ -invariant measures on  $W$  and  $f$ -invariant measures on  $X$  and their respective entropies – the so called Abramov formula (see Zweimüller [Zwe04]; for related results see Keller [Kel89], Bruin [Bru95]).

We give sufficient conditions on the potential function  $\varphi$  under which the above procedure can be applied guaranteeing the existence and uniqueness of equilibrium measures. Let us stress that we do this without resorting to the study of analyticity of the pressure function (see Remark 3 for more details). Results by Ruelle [Rue78] (see also Aaronson [Aar97]) describe some ergodic properties of equilibrium measures for the induced system (including exponential decay of correlations and the Central Limit Theorem) which, by Young [You98], can be transferred to the original system.

We stress that the thermodynamical formalism presented here depends on the choice of the inducing scheme, since the latter determines a class of  $f$ -invariant measures and a class of potential functions to which our theory applies. Note that one can construct different inducing schemes for a given map. Naturally one would like the class of measures and potentials, allowed by the scheme, to be as large as possible and, ideally, it should include all  $f$ -invariant measures and significant potential functions such as constants and  $\varphi_t = -t \log |df|$ .

In particular, our methods apply to transverse one-parameter families of unimodal maps with parameters from a set of positive Lebesgue measure, where the measures we consider have integrable inducing time and the class of potential functions includes  $\varphi_t = -t \log |df|$  with  $t$  in an interval containing  $[0, 1]$  (see [PS05]); this extends results by Bruin and Keller [BK98] for the parameters and measures under consideration). In particular, this gives a new and unified approach for establishing existence and uniqueness of measures of maximal entropy (first, constructed by Hofbauer [Hof79, Hof81]) as well as absolutely continuous invariant measures. Our results can also be used to observe phase transitions for some potentials associated with uniformly expanding maps of intervals (see [PZ05]).

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## 2. INDUCING SCHEMES

Let  $f : I \rightarrow I$  be a smooth map of a compact interval  $I$  and  $S$  a countable collection of open disjoint intervals. Let also  $\tau : S \rightarrow \mathbb{N}$  be a positive integer-valued function. Define the *inducing domain* by  $\mathcal{W} := \bigcup_{J \in S} J$ , the *inducing time*  $\tau : \mathcal{W} \rightarrow \mathbb{N}$  by

$$\tau(x) := \begin{cases} \tau(J), & x \in J \in S, \\ 0 & x \notin \mathcal{W}, \end{cases}$$

and the *induced map*  $F : \mathcal{W} \rightarrow I$  by  $F(x) = f^{\tau(x)}(x)$ . Denote

$$W := \bigcap_{n \geq 0} F^{-n}(\mathcal{W}), \quad X := \bigcup_{J \in S} \bigcup_{k=0}^{\tau(J)-1} f^k(W \cap J).$$

The set  $W$  is totally invariant under  $F$  and the set  $X$  is forward invariant under  $f$ . We call  $X$  the *tower* with the *base*  $W$ . In general,  $W$  may be an empty set. However, in many interesting cases including some unimodal maps (see Section 5) one can construct inducing schemes such that for an interval  $I'$  with  $W \subset \mathcal{W} \subset I' \subset I$  the Lebesgue measure of the set  $I' \setminus W$  is zero.

Let  $\bar{J}$  denote the closure of the set  $J$ . We say that  $f$  admits an *inducing scheme*  $\{S, \tau\}$  if:

- (H1)  $f^{\tau(J)}|_{\bar{J}}$  is a homeomorphism onto its image and  $f^{\tau(J)}(\bar{J}) \supseteq \mathcal{W}$ ;
- (H2) the partition  $\mathcal{R}$  of  $\mathcal{W}$  induced by the sets  $J \in S$  is generating, i.e., for any countable collection of intervals  $J_1, J_2, \dots \in S$  the intersection  $\bar{J}_1 \cap f^{-\tau(J_1)}(\bar{J}_2) \cap f^{-\tau(J_1)-\tau(J_2)}(\bar{J}_3) \cap \dots$  consists of a single point.

Condition (H2) often comes as a result of the fact that the induced map is expanding: there exists  $\lambda > 1$  with  $|dF(x)| > \lambda$  for every  $x \in \mathcal{W}$ .

The partition  $\mathcal{R}$  induces a partition of  $W$  which we denote by the same letter. Conditions (H1), (H2) allow one to obtain a symbolic representation of the induced map as the (full) shift  $\sigma$  on the space  $S^{\mathbb{N}}$  where  $S$  is a countable set of states. More precisely, let  $\tilde{W} = \bigcup_{J \in S} \bar{J}$  and define the *coding map*  $h : S^{\mathbb{N}} \rightarrow \tilde{W}$  by  $h((a_k)_{k \in \mathbb{N}}) = x$  where  $x$  is

such that  $x \in \bar{J}_{a_0}$  and

$$f^{\tau(J_{a_k})} \circ \dots \circ f^{\tau(J_{a_0})}(x) \in \bar{J}_{a_{k+1}} \quad \text{for } k \geq 0.$$

**Proposition 2.1.** *The map  $h$  is well-defined and is onto. It is one-to-one on  $h^{-1}(W)$  and is a topological conjugacy between  $\sigma|_{h^{-1}(W)}$  and  $F|_W$ . Moreover, the set  $S^{\mathbb{N}} \setminus h^{-1}(W)$  is countable.*

Let  $\mathcal{M}(F)$  denote the set of  $F$ -invariant ergodic Borel probability measures on  $W$  and  $\mathcal{M}(f)$  the set of  $f$ -invariant ergodic Borel probability measures on  $X$ . For any  $\nu \in \mathcal{M}(F)$  let

$$Q_\nu := \sum_{J \in S} \tau(J) \nu(J).$$

If  $Q_\nu < \infty$  we define the *lifted measure*  $\pi(\nu)$  on  $I$  in the following way (see for example [Zwe04]): for any measurable set  $E \subseteq I$

$$\pi(\nu)(E) := \frac{1}{Q_\nu} \sum_{J \in S} \sum_{k=0}^{\tau(J)-1} \nu(f^{-k}(E) \cap J).$$

Note that  $\pi(\nu) \in \mathcal{M}(f)$ . If  $Q_\nu = \infty$ , the measure given by

$$\sum_{J \in S} \sum_{k=0}^{\tau(J)-1} \nu(f^{-k}(E) \cap J),$$

is  $\sigma$ -finite but not finite.

We call a measure  $\mu \in \mathcal{M}(f)$  *liftable* if there is a measure  $\nu \in \mathcal{M}(F)$  such that  $\pi(\nu) = \mu$ . By definition,  $Q_\nu < \infty$  and as can be easily seen,  $\mu|_W \ll \nu$ . It follows that  $\nu$  is uniquely defined. We call  $\nu$  the *induced measure* for  $\mu$  and we write  $\nu = i(\mu)$ . Finally, we denote the set of liftable measures by  $\mathcal{M}^*(f)$ .

We also have the following result.

**Proposition 2.2** (Zweimüller [Zwe04]). *Any measure  $\mu \in \mathcal{M}(f)$  with  $\tau \in L^1(X, \mu)$  is liftable.*

For  $\varphi : I \rightarrow \mathbb{R}$  we define the *induced function*  $\tilde{\varphi} : \mathcal{W} \rightarrow \mathbb{R}$  by

$$\tilde{\varphi}(x) := \sum_{k=0}^{\tau(J)-1} \varphi(f^k(x)) \quad \text{for } x \in J.$$

Observe that given an interval  $J \in S$ , the function  $\tilde{\varphi}$  can be extended to a continuous function on  $\bar{J}$  which we still denote by  $\tilde{\varphi}$ . This extension is well-defined for all  $x \in J$ , but is "multi-valued" on the (countable) set of points  $x \in \bar{J} \cap \bar{J}'$ ,  $J \neq J' \in S$ , and the value of  $\tilde{\varphi}$  depends on the extension. In what follows the extension that should be used to

determine the values of the function at the endpoints of intervals will be clear from the context.

**Proposition 2.3** (Abramov's and Kac's Formulæ, see [Zwe04]). *If  $\mu \in \mathcal{M}^*(f)$  then*

$$h_{i(\mu)}(F) = Q_{i(\mu)} \cdot h_\mu(f) < \infty.$$

*If  $\int_X \varphi d\mu$  is finite then*

$$-\infty < \int_W \tilde{\varphi} di(\mu) = Q_{i(\mu)} \cdot \int_X \varphi d\mu < \infty.$$

### 3. EQUILIBRIUM MEASURES FOR THE INDUCED MAP

Consider the full shift  $\sigma$  on the set  $S^\mathbb{N}$ , where  $S$  is a countable set of states, and a potential function  $\Phi : S^\mathbb{N} \rightarrow \mathbb{R}$ .

The *Gurevich pressure* of  $\Phi$  is defined by (see for example, [Sar99]):

$$(1) \quad P_G(\Phi) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\sigma^n(\omega) = \omega} \exp(\Phi_n(\omega)) \chi_{[a]}(\omega),$$

where  $a \in S$ ,  $\chi_{[a]}$  is the characteristic function of the cylinder  $[a]$  and

$$\Phi_n(\omega) = \sum_{k=0}^{n-1} \Phi(\sigma^k(\omega)).$$

The *n-variation*  $V_n(\Phi)$  is defined by

$$V_n(\Phi) := \sup_{[b_0, \dots, b_{n-1}]} \sup_{\omega, \omega' \in [b_0, \dots, b_{n-1}]} \{|\Phi(\omega) - \Phi(\omega')|\}.$$

If  $\sum_{n \geq 1} V_n(\Phi) < \infty$  then the limit in (1) exists and does not depend on  $a \in S$  (see [Sar03]).

We call a measure  $\nu_\Phi$  a *Gibbs measure* for  $\Phi$  if there exist constants  $C_1 > 0$  and  $C_2 > 0$  such that for any cylinder  $[b_0, \dots, b_{n-1}]$  and any  $\omega \in [b_0, \dots, b_{n-1}]$  we have

$$C_1 \leq \frac{\nu_\Phi([b_0, \dots, b_{n-1}])}{\exp(-nP_G(\Phi) + \Phi_n(\omega))} \leq C_2.$$

**Proposition 3.1** ([Sar99], see also [MU01], [Aar97], [ADU93], [BS03]). *Assume that*

- $\Phi$  is continuous and  $\sup_{\omega \in S^\mathbb{N}} \Phi < \infty$ ,
- $P_G(\Phi) < \infty$ ,
- $\sum_{n \geq 1} V_n(\Phi) < \infty$ .

*Then there exists an ergodic  $\sigma$ -invariant Gibbs measure  $\nu_\Phi$  for  $\Phi$ .*

Observe that a Gibbs measure  $\nu_\Phi$  is positive on every non-empty open set and ergodic and hence, by Proposition 2.1,  $\nu_\Phi(h^{-1}(W)) = 1$ .

Given a *potential* function  $\varphi : I \rightarrow \mathbb{R}$  and its induced potential  $\tilde{\varphi}$ , we denote by

$$\mathcal{M}_{\tilde{\varphi}}(F) := \{\nu \in \mathcal{M}(F) : - \int_W \tilde{\varphi} d\nu < \infty\}.$$

We call a measure  $\nu_{\tilde{\varphi}} \in \mathcal{M}_{\tilde{\varphi}}(F)$  an *equilibrium measure* for  $\tilde{\varphi}$  (with respect to the class of measures  $\mathcal{M}_{\tilde{\varphi}}(F)$ ) if

$$\sup_{\nu \in \mathcal{M}_{\tilde{\varphi}}(F)} \{h_\nu(F) + \int_W \tilde{\varphi} d\nu\} = h_{\nu_{\tilde{\varphi}}}(F) + \int_W \tilde{\varphi} d\nu_{\tilde{\varphi}}.$$

Given a potential function  $\varphi : I \rightarrow \mathbb{R}$ , define the potential function  $\Phi : S^\mathbb{N} \rightarrow \mathbb{R}$  as follows:

$$\Phi(\omega) := \tilde{\varphi} \circ h(\omega) = \sum_{i=0}^{\tau(J_0)-1} \varphi(f^i(h(\omega))) \quad \text{if } \omega = (J_0, J_1, \dots) \in S^\mathbb{N}$$

(recall that  $\tilde{\varphi}$  is extended on each  $J \in S$  to its closure  $\bar{J}$ ). Note that  $\Phi$  is continuous in the discrete topology of  $S^\mathbb{N}$ .

The following result establishes existence and uniqueness of equilibrium measures for the induced map and a certain class of potential functions.

**Theorem 3.2.** *Assume that  $\Phi$  satisfies the conditions of Proposition 3.1. If, in addition, the entropy  $h_{\nu_\Phi}(\sigma) < \infty$ , then the measure*

$$\nu_{\tilde{\varphi}} := h_* \nu_\Phi$$

*belongs to  $\mathcal{M}_{\tilde{\varphi}}(F)$  and it is the unique equilibrium measure for  $\tilde{\varphi}$  (with respect to the class of measures  $\mathcal{M}_{\tilde{\varphi}}(F)$ ).*

The proof follows from Propositions 2.1 and 3.1 and the fact that  $\nu_\Phi(h^{-1}(W)) = 1$ .

#### 4. EXISTENCE, UNIQUENESS AND ERGODIC PROPERTIES OF EQUILIBRIUM MEASURES

Denote by

$$s_\varphi := \sup_{\mu \in \mathcal{M}^*(f)} \left( h_\mu(f) + \int_X \varphi d\mu \right).$$

A measure  $\mu \in \mathcal{M}^*(f)$  for which this supremum is attained is called an *equilibrium measure* for  $\varphi$  (with respect to the class of measures  $\mathcal{M}^*(f)$ ).

Our definitions of equilibrium measures differ from the classical ones. First, we only consider the supremum over measures for which  $W$  (respectively,  $X$ ) is of full measure. Second, we only allow measures  $\mu \in \mathcal{M}^*(f)$ , i.e., liftable measures. For a general inducing scheme one may not be able to drop these restrictions: Pesin and Zhang [PZ05] gave an example of a one-dimensional map  $f$  with an inducing scheme and of a potential  $\varphi$  for which the supremum over all  $f$ -invariant ergodic Borel probability measures supported on the closure  $\bar{X}$  of the tower  $X$  is strictly bigger than the one taken over  $\mu \in \mathcal{M}(f)$  (the former is attained by a measure with  $\mu(X) = 0$ ). Also Zhang (following an example of Zweimüller [Zwe04]) constructed an example of an abstract tower for which the supremum over all measures  $\mu \in \mathcal{M}(f)$  is attained by a measure with  $\int \tau(x) d\mu = \infty$  and is strictly larger than the one taken over measures with integrable inducing time (unpublished).

Given a potential function  $\varphi : X \rightarrow \mathbb{R}$ , we denote by  $\phi^+$  the induced potential of the function  $\varphi - s_\varphi$ . Observe that  $\phi^+(x) = \tilde{\varphi}(x) - s_\varphi \tau(x)$ .

The class of potential functions  $\varphi$  is defined by a collection of axioms that guarantee that the conditions of Theorem 3.1 hold and that the lifted measure of an equilibrium measure for the induced system is an equilibrium measure for the original system.

More precisely, we assume the following conditions on  $\varphi$ :

- (P1) (*boundedness*):  $\sup_{x \in W} \phi^+(x) < \infty$ ;
- (P2) (*local Hölder continuity*):  $\varphi$  is continuous on  $W$  and there exist  $A > 0$  and  $0 < r < 1$  such that  $V_n(\tilde{\varphi} \circ h) \leq Ar^n$  for all  $n \geq 1$ ;
- (P3) (*finite Gurevich pressure*):

$$\sum_{J \in S} \sup_{x \in J} \exp \tilde{\varphi}(x) < \infty;$$

- (P4) (*positive recurrence*): there exists  $\varepsilon_0 > 0$  such that for every  $0 \leq \varepsilon \leq \varepsilon_0$ ,

$$\sum_{J \in S} \sup_{x \in J} \exp(\phi^+(x) + \varepsilon \tau(x)) < \infty.$$

The following statement establishes existence and uniqueness of equilibrium measures for the map  $f$ .

**Theorem 4.1.** *Let  $f$  be a smooth one-dimensional map of a compact interval admitting an inducing scheme  $\{S, \tau\}$ . Let also  $\varphi$  be a potential function satisfying Conditions (P1) – (P4). Then there exists a unique equilibrium measure  $\mu_\varphi \in \mathcal{M}^*(f)$  for  $\varphi$  (with respect to the class of measures  $\mathcal{M}^*(f)$ ).*

We outline the proof of the theorem (for the complete proof see [PS05]). By Conditions (P2) and (P3) respectively, the induced potential function  $\tilde{\varphi}$  has summable variations and finite Gurevich pressure. This implies that  $s_\varphi$  is finite. By Conditions (P1), (P2) and (P4) (with  $\varepsilon = 0$ ), the induced potential  $\phi^+$  corresponding to the “normalized” potential  $\varphi - s_\varphi$  is bounded from above, has summable variations and finite Gurevich pressure. Therefore, by Proposition 3.1, there exists a unique Gibbs measure  $\nu_{\phi^+} := h_*\nu_{\Phi^+}$  for  $\phi^+$  on  $W$ , where  $\Phi^+ := \phi^+ \circ h$ . Condition (P4) implies that

$$\sum_{J \in S} \tau(J) \sup_{x \in J} \exp(\phi^+(x)) < \infty.$$

One can show that this yields  $Q_{\nu_{\phi^+}} < \infty$  and hence,  $-\int_W \phi^+ d\nu_{\phi^+} < \infty$ . It follows that  $\nu_{\phi^+} \in \mathcal{M}_{\phi^+}(F)$  and it is an equilibrium measure for  $\phi^+$ . To prove that this natural candidate is indeed an equilibrium measure for  $\varphi$  one needs to verify the *recurrence property*:  $P_G(\phi^+) = 0$  (see for example [Sar99]). This can be done using Condition (P4). The variational principle now implies that

$$h_{\nu_{\phi^+}}(F) + \int_W \phi^+ d\nu_{\phi^+} = 0.$$

By Abramov’s and Kac’s formulas (see Proposition 2.3), we have that

$$Q_{\nu_{\phi^+}}(h_{\pi(\nu_{\phi^+})}(f) + \int_I (\varphi - s_\varphi) d\pi(\nu_{\phi^+})) = 0$$

since  $Q_{\nu_{\phi^+}} < \infty$ , and the desired result follows.

In order to study the ergodic properties of the equilibrium measure, we introduce the following additional condition:

(P5) (*exponential tail*): there exist  $K > 0$  and  $0 < \theta < 1$  such that for all  $n > 0$ ,

$$\nu_{\phi^+}(\{x \in W : \tau(x) \geq n\}) \leq K\theta^n.$$

**Theorem 4.2.** *Under the same hypothesis as in Theorem 4.1 and Condition (P5), the equilibrium measure  $\mu_\varphi$  has exponential decay of correlations and satisfies the Central Limit Theorem for the class of functions whose induced function are Hölder continuous on  $W$ .*

By Theorem 4.1, the equilibrium measure  $\nu_{\phi^+}$  exists and by results of Ruelle [Rue78] (see also Aaronson [Aar97]), it has exponential decay of correlations and satisfies the Central Limit Theorem for the class of Hölder continuous potentials on  $W$ . By results of Young [You98], under Condition (P5), these ergodic properties can be transferred to the equilibrium measure  $\mu_\varphi$ .



**Remark 1.** We call two functions  $\varphi$  and  $\psi$  *cohomologous* if there exists a bounded function  $h$  and a real number  $C$  such that  $\varphi - \psi = h \circ f - h + C$ . Any equilibrium measure for  $\varphi$  is an equilibrium measure for any  $\psi$  cohomologous to  $\varphi$  and conversely. In particular, if  $\varphi$  satisfies Conditions (P1)–(P4) then there is a unique equilibrium measure for any  $\psi$  cohomologous to  $\varphi$  regardless of whether this function satisfies our Conditions (P1)–(P4).

**Remark 2.** Unlike the classical thermodynamical formalism our approach to establish existence and uniqueness of equilibrium measures does not directly rely on analyticity of the *pressure function*  $t \rightarrow P_G(\tilde{\varphi}_1 + t\tilde{\varphi}_2)$  where  $\tilde{\varphi}_1$  and  $\tilde{\varphi}_2$  are the induced potentials for some given potential functions  $\varphi_1$  and  $\varphi_2$  respectively. On the other hand, using results of Sarig [Sar01], one can show that, if the function  $\varphi_1$  satisfies Conditions (P2) and (P4), if the function  $\varphi_2$  satisfies Condition (P2) and if the function  $\varphi_1 - s_{\varphi_1} + t\varphi_2$  satisfies Condition (P3) for  $t$  near 0, then  $P_G(\varphi_1 - s_{\varphi_1} + t\varphi_2)$  is analytic in  $t$ .

## 5. EQUILIBRIUM MEASURES FOR UNIMODAL MAPS

We describe applications of our results to families of unimodal maps.

Let  $f$  be a  $C^3$  interval map with one non-flat critical point belonging to the interior of the interval of definition of  $f$ . Without loss of generality, we may assume that 0 is the critical point and that  $f$  is symmetric with respect to 0, so  $f : [-b, b] \rightarrow [-b, b]$  for some  $b > 0$ . Assume further that  $f(x) = \pm|\theta(x)|^l + f(0)$  for some local  $C^1$  diffeomorphism  $\theta$  and  $l > 1$ . The map  $f$  is *unimodal* if the derivative  $df/dx$  changes signs at 0 and  $f(\pm b) \in \{\pm b\}$ . Results for a unimodal map  $f$  customarily require that  $f$  has negative Schwarzian derivative, but this condition can be dropped if  $f$  has no attracting periodic points (see [GS05]).

A smooth one-parameter family of unimodal maps  $f_a$  is called *transverse* in a neighborhood of a parameter  $a^*$  if

$$\frac{d}{da}f_a(0) \neq \frac{d}{da}\zeta(a),$$

where  $\zeta(a)$  is the smooth continuation of the point  $x^* := \zeta(a^*) := f_{a^*}(0)$ . For any such family, Yoccoz [Yoc97] (see also [Sen03]) introduced the set  $\mathcal{A}$  of *strongly regular* parameters, which has positive Lebesgue measure (such parameters also satisfy the Collet-Eckmann condition).

Fix  $a \in \mathcal{A}$  and set  $f = f_a$ . Let  $I_{[b_0, \dots, b_{n-1}]}$  denote the set of points  $x$  with  $F^i(x) \in I_{[b_i]}$  for  $0 \leq i \leq n-1$  where  $F$  denotes the induced map for  $f$ .

**Theorem 5.1** ([Yoc97],[Sen03],[PS05]). *Consider a transverse one-parameter family of unimodal maps with no attracting periodic points. There exists a positive Lebesgue measure set  $\mathcal{A}$  of parameters such that for every  $a \in \mathcal{A}$  the map  $f = f_a$  admits an inducing scheme  $\{S, \tau\}$  satisfying (H1), (H2). In addition, there exists an interval  $I'$  with  $W \subset I' \subset I$  such that  $\text{Leb}(I' \setminus W) = 0$  and the following conditions hold:*

(H3) *there are constants  $c_1 > 0$  and  $\lambda_1 > 1$  such that for all  $n \geq 0$ ,*

$$\sum_{J \in S: \tau(J) \geq n} |J| \leq c_1^{-1} \lambda_1^{-n};$$

(H4) *(Bounded Distortion): for each  $n \geq 0$ , each interval  $I_{[b_0, \dots, b_{n-1}]}$  and each  $x, y \in I_{[b_0, \dots, b_{n-1}]}$ ,*

$$\left| \frac{dF^n(x)}{dF^n(y)} - 1 \right| \leq c_2 |F^n(x) - F^n(y)|;$$

(H5) *for every  $\gamma > 1$  there exists  $c = c(\gamma)$  such that*

$$\text{Card}\{J \in S \mid \tau(J) = n\} \leq c\gamma^n.$$

As a corollary of this theorem one has the following estimates: there exist positive constants  $c_2, c_3, c_4$  and  $\lambda_2 \geq \lambda_1$  such that for every  $J \in S$  and  $x \in J$ ,

$$(2) \quad c_1 \lambda_1^{\tau(J)} \leq |J|^{-1} \leq c_2 \lambda_2^{\tau(J)}, \quad c_3 |J|^{-1} \leq |dF(x)| \leq c_4 |J|^{-1}.$$

Given a map  $f_a$  from a transverse family of unimodal maps, we consider the potential function

$$\varphi_{a,t}(x) = -t \log |df_a(x)|.$$

If the parameter  $a$  is strongly regular, using (H3)–(H5) and (2), one can show that the function  $\varphi_{a,t}$  satisfies Conditions (P1)–(P4) and hence, results of the previous section apply.

**Theorem 5.2** ([PS05]). *Consider a transverse one-parameter family of unimodal maps with no attracting periodic points. There exists a positive Lebesgue measure set  $\mathcal{A}$  of parameters such that for every  $a \in \mathcal{A}$  the following statements hold.*

- (1) *There exist  $t_0(a) < 0 < 1 < t_1(a)$  such that for every  $t_0(a) < t < t_1(a)$  there is a measure  $\mu_{a,t}$  on  $I$  which is the unique equilibrium measure for  $\varphi_{a,t}$  (with respect to the class of measures  $\mathcal{M}^*(f)$ ). Moreover, for any measure  $\mu$  not supported on  $X$ ,*

$$h_\mu(f_a) - t \int \log |df_a(x)| d\mu < h_{\mu_{a,t}}(f_a) - t \int \log |df_a(x)| d\mu_{a,t}.$$

- (2) The measure  $\mu_{a,t}$  is ergodic, has exponential decay of correlations and satisfies the Central Limit Theorem for the class of functions whose induced functions are Hölder continuous. The measure  $\mu_{a,1}$  is the absolutely continuous invariant measure and the measure  $\mu_{a,0}$  is the unique measure of maximal entropy.

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